UDC 539.3

NECESSARY AND SUFFICIENT CONDITIONS FOR HOMOGENEOUS SIMPLE STRAIN

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G. L. BROVKO
(Moscow)

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The practice of computations shows that the use of the theory of small elastic-plastic strains yields completely satisfactory results for materials and loads of a broader class than is specified by conditions of the theorem on simple loading [1]. Necessary and sufficient conditions are obtained for homogeneous-simple strain for a broad class of compressible elastic-plastic solids with the dependence between the stress and strain intensities limited only by conditions on the ellipticity of the equilibrium boundary value problem. The form of the external loads under simple strain can hence differ substantially from their proportionate changes and, in general, depend on the mechanical characteristics of the material. These results afford a foundation for application of the theory of small elastic-plastic strains to a broader class of materials and loads.

1. Quasithermal fields. Lemma on linear superposition. Let us examine a continuous medium whose mechanical properties under isothermal conditions are governed by the relationship between the stress and strain deviators

$$s_{ij} = \Phi_{ij}(\partial_{kl}) \tag{1.1}$$

whose form is independent of the magnitude of the hydrostatic stress, and by the relationship for the global tensors

$$\sigma = 3K\varepsilon \tag{1.2}$$

Here K is the volume elastic modulus, and the functional Φ_{ij} is such that all $s_{ij} \equiv \sigma_{ij} - \sigma\delta_{ij} = 0$ if and only if all $\partial_{ij} = \varepsilon_{ij} - \varepsilon\delta_{ij} = 0$.

A sufficiently broad class of solids possesses such mechanical properties.

Let the medium mentioned fill a simply-connected domain Ω of the three-dimensional space bounded by a surface S. Let us examine the mixed boundary value problem of quasistatics when a loading process by the volume forces $F_i(\mathbf{x}, t)$, defined in the domain $Q = \Omega \times [0, T]$, the surface forces $T_{\nu i}(\mathbf{x}, t)$, defined on the part $\Sigma_\sigma = S_\sigma \times [0, T]$ of the side boundary $\Sigma = S \times [0, T]$ of the domain Q, and the displacements $\psi_i(\mathbf{x}, t)$ on the part $\Sigma_u = S_u \times [0, T]$ of the boundary Σ ($S_u + S_\sigma = S$, $\Sigma_u + \Sigma_\sigma = \Sigma$) is given for the body Ω in the time segment $t \in [0, T]$. The problem is to find displacement $u_i(\mathbf{x}, t)$, strain $\varepsilon_{ij}(\mathbf{x}, t)$, and stress $\sigma_{ij}(\mathbf{x}, t)$ functions which will satisfy the relationships (1.1) and (1.2) in the domain $Q' = Q + \Sigma$, the equilibrium equations (here and below the summation is over repeated Latin and not the Greek subscripts)

$$\sigma_{ij,j} + F_i = 0 \quad \text{in } Q \tag{1.3}$$

the Cauchy relations

$$\varepsilon_{ij} = {}^1/_2\,(u_{i,\,j} + u_{j,\,i}) \quad \text{in} \quad Q' \tag{1.4}$$

and also the boundary conditions

$$\sigma_{ij} l_j = T_{\nu i} \quad \text{on} \quad \Sigma_\sigma \tag{1.5}$$

$$u_i = \psi_i \quad \text{on} \quad \Sigma_u \tag{1.6}$$

where $l_i = l_i\,(\mathbf{x})$ are the direction cosines of the external normal to S.

Let us assume the problem (1.1)-(1.6) to be solvable in a unique way.

We find the class of loading processes for which the strain and stress tensors are global at all points of the body at any time of the process.

L e m m a *1.1.* In order for the solution of the boundary value problem (1.1) -(1.6) to represent an arbitrary pure volume strain process for the body, it is necessary and sufficient that the external loads have the form

$$\begin{aligned} &F_i^* = F_i^*\,(t) \quad \text{in } Q' \\ &T_{\nu i}^*\,(\mathbf{x},\,t) = -[F_j^*\,(t)\,x_j + D\,(t)]\,l_i \quad \text{on} \quad \Sigma_\sigma \\ &\psi_i^*\,(\mathbf{x},\,t) = (3K)^{-1}\,\{{}^1/_2\,F_i^*\,(t)\,(x_j x_j) - x_i\,[F_i\,(t)\,x_j + \\ &\quad D\,(t)]\} + a_i \quad \text{on} \quad \Sigma_u \end{aligned} \tag{1.7}$$

Here $F_i^*\,(t), D\,(t)$ are arbitrary functions on $[0,\,T]$, $a_i = a_i\,(\mathbf{x},\,t)$ is an arbitrary displacement vector for the body as a rigid whole.

We prove the necessity of these conditions. Let $\varepsilon_{ij}\,(\mathbf{x},\,t) = \varepsilon\,(\mathbf{x},\,t)\,\delta_{ij}$. Then from the strain compatibility equations

$$\begin{aligned} &\varepsilon_{\alpha\alpha,\,\beta\beta} + \varepsilon_{\beta\beta,\,\alpha\alpha} = 2\varepsilon_{\alpha\beta,\,\alpha\beta} \\ &\varepsilon_{\alpha\alpha,\,\beta\gamma} = (-\,\varepsilon_{\beta\gamma,\,\alpha} + \varepsilon_{\gamma\alpha,\,\beta} + \varepsilon_{\alpha\beta,\,\gamma})_{,\,\alpha} \\ &(\alpha,\,\beta,\,\gamma = 1,\,2,\,3;\ \alpha \neq \beta \neq \gamma \neq \alpha) \end{aligned} \tag{1.8}$$

there follows $\varepsilon_{,\alpha\beta} = 0$ $(\alpha,\ \beta = 1, 2, 3)$, i.e.,

$$\varepsilon_{ij}\ (\mathbf{x},\ t) = [C_k\,(t)\,x_k + C_0\,(t)]\,\delta_{ij} \quad \text{in} \quad Q' \tag{1.9}$$

where $C_k\,(t)$ and $C_0\,(t)$ are arbitrary functions on $[0,\ T]$.

It follows from the properties of the functional (1.1) that the stress tensor is also global $\sigma_{ij}\,(\mathbf{x},\,t) = \sigma\,(\mathbf{x},\,t)\delta_{ij}$, where according to (1.2) and (1.9) $\sigma\,(\mathbf{x},\,t) = 3K$ $[C_k\,(t)x_k + C_0\,(t)]$ i.e.,

$$\sigma_{ij}\,(\mathbf{x},\,t) = 3K\,[C_k\,(t)x_k + C_0\,(t)]\delta_{ij} \quad \text{in} \quad Q' \tag{1.10}$$

In this case the general solution of the Cauchy equations (1.4) is

$$u_i\,(\mathbf{x},\ t) = -{}^1/_2 C_i\,(t)\,(x_j x_j) + x_i\,[C_j\,(t)x_j + C_0\,(t)] + b_i \quad \text{in} \quad Q' \tag{1.11}$$

Here $b_i = b_i\,(\mathbf{x},\,t)$ is an arbitrary displacement vector of the body as a rigid whole.

In order for the functions (1.9)-(1.11) to satisfy the equilibrium equations (1.3) and the boundary conditions (1.4), it is necessary that

$$F_i = -3KC_i\,(t) \quad \text{in } Q' \tag{1.12}$$
$$T_{\nu i} = 3K\,[C_j\,(t)x_j + C_0\,(t)]l_i \quad \text{on } \Sigma_\sigma$$
$$\psi_i = -{}^1/_2 C_i\,(t)\,(x_j x_j) + x_i\,[C_j\,(t)x_j + C_0\,(t)] + b_i \quad \text{on } \Sigma_u$$

from which we establish the necessity of the conditions (1.7) by setting $F_i^*\,(t) = -3KC_i\,(t)$, $D\,(t) = -3KC_0\,(t)$ and $a_i\,(\mathbf{x}, t) = b_i\,(\mathbf{x}, t)$.

To prove the sufficiency of the conditions, it should just be noted that the functions

$$u_i^*\,(\mathbf{x}, t) = (3K)^{-1}\,\{{}^1/_2 F_i^*\,(t)\,(x_j x_j) - x_i\,[F_j^*\,(t)x_j + D(t)]\} + a_i \tag{1.13}$$
$$\varepsilon_{ij}^*\,(\mathbf{x}, t) = -(3K)^{-1}\,[F_k^*\,(t)x_k + D\,(t)]\delta_{ij}$$
$$\sigma_{ij}^*\,(\mathbf{x}, t) = -[F_k^*\,(t)x_k + D\,(t)]\delta_{ij}$$

in Q' are the solution of the problem (1.1)-(1.6) under the loads (1.7), which is unique by assumption.

Therefore, the process of pure volume inhomogeneous strain, described for a body from materials with the properties (1.1) and (1.2), is characterized by four independent scalar functions $C_i\,(t)$ $(i = 0, 1, 2, 3)$ and corresponds to an arbitrary change in time of the homogeneous field of volume forces $\mathbf{F}(t)$ for inhomogeneous loads on the surface defined according to (1.12). The integral equilibrium conditions expressed by the functions (1.12) for the body are satisfied identically and impose no limitations (constraints) on the functions $C_i\,(t)$.

Lemma 1.1 allows the following interpretation: if a homogeneous body with the mechanical properties (1.1) and (1.2) floats in equilibrium in a homogeneous gravity fluid in a homogeneous field of mass forces, then it experiences just volume strains and stresses.

Let us call the field of external loads F_i^*, $T_{\nu i}^*$, ψ_i^* defined by (1.7) quasithermal in application to the problems of the type (1.1)-(1.6), and the solution (1.13), corresponding to this field, the quasithermal solution.

Lemma *1.2* about linear superposition. For given F_i, $T_{\nu i}$, ψ_i let the boundary value problem (1.1)-(1.6) have the solution u_i, ε_{ij}, σ_{ij}. Then upon the imposition of an arbitrary quasithermal field of loads (1.7) the solution is obtained by a linear superposition of the corresponding quasithermal solution (1.13), i.e., for the loads $F_i + F_i^*$, $T_{\nu i} + T_{\nu i}^*$, $\psi_i + \psi_i^*$ the solution of the problem (1.1)-(1.6) is $u_i + u_i^*$, $\varepsilon_{ij} + \varepsilon_{ij}^*$, $\sigma_{ij} + \sigma_{ij}^*$.

The proof follows from Lemma 1.1 the linearity of the relationships (1.2)-(1.6), and the independence of the nonlinear relationships (1.1) from the volume strains and stresses.

It is seen that the set of quasithermal process $\{F_i^*, T_{\nu i}^*, \psi_i^*, u_i^*, \varepsilon_{ij}^*, \sigma_{ij}^*\}$ is a linear space, and the whole set of processes $\{F_i, T_{\nu i}, \psi_i, u_i, \varepsilon_{ij}, \sigma_{ij}\}$ which are subject to the relationships (1.1)-(1.6), is factorized by Lemma 1.2 in the subspace of quasithermal processes. Here the process should be understood to be the vector function $\{F_i, T_{\nu i}, \psi_i, u_i, \varepsilon_{ij}, \sigma_{ij}\}$ defined in $Q \times \Sigma_\sigma \times \Sigma_u \times Q' \times Q' \times Q'$ and comprised of components of the external load field and the solution corresponding to this field.

Finally, let us note that the sum of any tensor satisfying the strain compatibility equations (1.8) and the arbitrary quasithermal strain tensor (1.9) also satisfied (1.8).

2. Homogeneous-simple strain. Kinematics and representation of the process. The process of body strain is called simple [1] (simple according to the deviator) if the field of the directional strain tensor remains invariant during the whole process, i.e.,

$$\vartheta_{ij}(\mathrm{x}, t) = A\vartheta_{ij}^\circ(\mathrm{x}) \quad \text{in } Q' \tag{2.1}$$

Here $\vartheta_{ij}^\circ(\mathrm{x})$ is some deviator given in Ω', and $A = A(\mathrm{x}, t)$ is a certain function of the coordinate and time. The relationship (2.1) can be written in the form

$$p_{ij} = p_{ij}(\mathrm{x}), \quad \vartheta_u(\mathrm{x}, t) = A\vartheta_u^\circ(\mathrm{x}) \tag{2.2}$$

Here p_{ij} is the time-invariant directional strain tensor $\vartheta_u = ({}^2/_3\vartheta_{ij}\vartheta_{ij})^{1/2}$ is the strain intensity of the process, and ϑ_u° is the intensity of the deviator ϑ_{ij}°.

We call the body strain process homogeneously simple if $A \equiv A(t)$. Only such simple strain processes are considered below.

The value of the deviator of the process at any fixed time not identically zero, or any value proportional to this can be taken as the generating deviator $\vartheta_{ij}^\circ(x)$ of the simple strain process in (2.1). Every other method of selecting the generating deviator reduces to this. In particular, the proposition: For any linear operator L in a time independent of x, and any t_0 and t_1 from $[0, T]$ $(A(t_1) \neq 0)$, there is a number $C = C(L, t_0, t_1)$ such that

$$L[\vartheta_{ij}(\mathrm{x}, t)]_{t=t_0} = C\vartheta_{ij}(\mathrm{x}, t_1) \tag{2.3}$$

is valid for an arbitrary homogeneous strain process.

Here and henceforth, the time should be understood to be any parameter distinguishing a sequence of events, i.e. any increasing scalar function of the physical time $\lambda(t)$.

Indeed, we obtain from (2.1)

$$L[\vartheta_{ij}(\mathrm{x}, t)]_{t=t_0} = L[A(t)\vartheta_{ij}^\circ(\mathrm{x})]_{t=t_0} = L[A(t)]_{t=t_0} \times \vartheta_{ij}^\circ(\mathrm{x}) = L[A(t)]_{t=t_0}A^{-1}(t_1)\vartheta_{ij}(\mathrm{x}, t_1) = C\vartheta_{ij}(\mathrm{x}, t_1), \quad C = L[A(t)]_{t=t_0}A^{-1}(t_1)$$

Therefore, if $C \neq 0$ in the relationship (2.3) then

$$\vartheta_{ij}^\circ(\mathrm{x}) = L[\vartheta_{ij}(\mathrm{x}, t)]_{t=t_0} \tag{2.4}$$

can be taken as the generating deviator of the process (2.1).

It is easy to see that for each separate process (2.1) with a change in the selection of $\vartheta_{ij}^\circ(\mathrm{x})$ the function $A(t)$ varies proportionately to itself.

Let us consider the kinematics of the homogeneous simple strain processes of compressible media filling an arbitrary simply-connected domain in three-dimensional space. Taking account of (2.1), we represent the strain tensor $\varepsilon_{ij} \equiv \partial_{ij} + \varepsilon\delta_{ij}$ in the form

$$\varepsilon_{ij}(\mathrm{x}, t) = A(t)\varepsilon_{ij}^{\circ}(\mathrm{x}) + \varepsilon^*(x, t)\delta_{ij} \tag{2.5}$$

Here $\varepsilon_{ij}^{\circ}(\mathrm{x}) \equiv \partial_{ij}^{\circ}(\mathrm{x}) + \varepsilon^{\circ}(\mathrm{x})\delta_{ij}$ is a tensor whose deviator part agrees with the generating deviator $\partial_{ij}^{\circ}(\mathrm{x})$ of the process, i.e., ε_{ij}° is defined in conformity with (2.4) to the accuracy of an arbitrary global tensor by the relationship

$$\varepsilon_{ij}^{\circ}(\mathrm{x}) = L[\varepsilon_{ij}(\mathrm{x}, t)]_{t=t_0} \tag{2.6}$$

The quantity $\varepsilon^*(\mathrm{x}, t) \equiv \varepsilon(\mathrm{x}, t) - A(t)\varepsilon^{\circ}(\mathrm{x})$ describes the deviation of the mean strain of the process ε from the proportionately varying $A\varepsilon^{\circ}$.

We shall henceforth assume that the strain tensor $\varepsilon_{ij}(\mathrm{x}, t)$ satisfies the compatibility equations (1.8) during the whole process. Then by virtue of the linearity of the operator L in (2.6) and the linearity of equations (1.8), the tensor $\varepsilon_{ij}^{\circ}(\mathrm{x})$ also satisfies (1.8). Taking this into account, we obtain $\varepsilon^*_{,\alpha\beta} = 0$ in Q' $(\alpha, \beta = 1, 2, 3)$ from (2.5) and (1.8), from which we conclude analogously to (1.9) that the tensor $\varepsilon^*(\mathrm{x}, t)\delta_{ij}$ defines a process of purely volume (quasithermal) strain of a body and together with the displacement vector corresponding to this process $u_i^*(\mathrm{x}, t)$, is expressed by (1.9) and (1.11), respectively. If the combined tensor $\varepsilon_{ij}^{\circ}(\mathrm{x})$ is selected in general form corresponding to (2.4), i.e., in the form of the sum of (2.6) and the arbitrary combined global tensor of the form (1.9), we arrive at the same result.

Therefore, the global strain tensor for homogeneous-simple strain varies proportionally to the deviator (the function $A(t)$) to the accuracy of the quasithermal strain tensor

$$\varepsilon(\mathrm{x}, t)\delta_{ij} = A(t)\varepsilon^{\circ}(\mathrm{x})\delta_{ij} + \varepsilon^*(\mathrm{x}, t)\delta_{ij}$$

The expression corresponding to (2.5) for the displacement vector of the homogeneous simple strain process has the form

$$u_i(\mathrm{x}, t) = A(t)u_i^{\circ}(\mathrm{x}) + u_i^*(\mathrm{x}, t) \tag{2.7}$$

Here the vector $u_i^{\circ}(\mathrm{x})$ is the solution of the Cauchy equation $\varepsilon_{ij}^{\circ} = {}^1/_2(u_{ij}^{\circ} + u_{j,i}^{\circ})$ and is defined in case of (2.6) to the accuracy of an arbitrary displacement of the body as a rigid whole by the relationship

$$u_i^{\circ}(\mathrm{x}) = L[u_i(\mathrm{x}, t)]_{t=t_0} \tag{2.8}$$

All arbitrary rigid body displacements occurring in the resolution of the Cauchy equatiions are included in the expression for $u_i^*(\mathrm{x}, t)$. The first members in the right sides of (2.5) and (2.7) evidently govern the tensor-simple strain process.

L e m m a *2.1*. Kinematically the homogeneous-simple (deviator simple) strain process of simply-connected compressible bodies is a linear superposition of two independent processes: 1) the tensor-simple strain process, and the pure volume (quasithermal) strain process for a body.

In conformity with this, the set of kinematic representations (2.5) and (2.7) of all deviator-simple strain processes is factorized in the linear subspace of kinematic representations (1.9) and (1.11) of arbitrary quasithermal strain processes.

This result can be extended to the state of stress of a body, as well as to the loading process during deviator simple strain, in applications to problems of the type (1.1)-(1.6) on the basis of Lemma 1.2.

Corollary *2.1*. Every deviator-simple strain process can be represented in the form of the sum

$$\{F_i,\ T_{\nu i},\ \psi_i,\ u_i,\ \varepsilon_{ij},\ \sigma_{ij}\} = \{F_i{}^t + F_i{}^*,\ T_{\nu i}{}^t + T_{\nu i}{}^*,\ \psi_i{}^t + \psi_i{}^*,\ u_i{}^t + u_i{}^*,\ \varepsilon_{ij}{}^t + \varepsilon_{ij}{}^*,\ \sigma_{ij}{}^t + \sigma_{ij}{}^*\}$$

Here the superscript ^{t} denotes functions describing the tensor-simple strain process, and the superscript $*$ the quasithermal process. Therefore, the diviator-simple, and tensor-simple strain processes having identical strain deviators, drop into one equivalence class during factorization of a set of processes of the type (1.1)-(1.6) in the subspace of quasithermal processes. Hence, the question of the homogeneous simple strain conditions reduces to the determination of the tensor-simple strain conditions described by the relations

$$\varepsilon_{ij}(\mathrm{x},t) = A(t)\varepsilon_{ij}{}^\circ(\mathrm{x}),\quad u_i(\mathrm{x},t) = A(t)u_i{}^\circ(\mathrm{x}) \qquad (2.9)$$

obtained from (2.5) and (2.7) by discarding the quasithermal terms.

3. Initially and infinitesimally - elastic materials. Theorem of homogeneous simple strain. Let us turn to an investigation of the homogeneous simple strain conditions in solids. Limiting ourselves to the subclass of tensor-linear plasticity theories described by (1.1) and (1.2), and taking account of the agreement between tensor-linear theories and theories of small elastic-plastic strains if the strain is simple [1], let us seek the formulation of these conditions in the terminology of the theory of small elastic-plastic strains.

The quasistatics boundary value problem of the theory of small elastic-plastic strains under active strain is described analogously to problem (1.1)-(1.6) by the relationships [1, 2]

$$\begin{gathered}\sigma_{ij,i} + F_i = 0 \quad \text{in } Q \\ s_{ij} = (2\sigma_u/3\vartheta_u)\vartheta_{ij},\quad \sigma_u = 3G\vartheta_u[1-\omega(\vartheta_u)], \\ \sigma = 3K\varepsilon,\quad \varepsilon_{ij} = {}^1/_2(u_{i,j}+u_{j,i}) \text{ in } Q' \\ \sigma_{ij}l_j = T_{\nu i} \quad \text{on } \Sigma\sigma \\ u_i = \psi_i \quad \text{on } \Sigma_u\end{gathered} \qquad (3.1)$$

Here the notation has the same meaning as in the problem (1.1)-(1.6); G is the initial shear modulus of the material, which is assumed positive, i.e.,

$$3G = \lim_{\vartheta_u \to 0} \frac{\sigma_u}{\vartheta_u} > 0 \tag{3.2}$$

The function $\omega(\vartheta_u)$ characterizes the plastic shear properties of the material [1]. For the majority of known materials, the function $\omega(\vartheta_u)$ satisfies the inequalities

$$0 \leqslant \omega(\vartheta_u) \leqslant \omega(\vartheta_u) + \vartheta_u \omega'(\vartheta_u) \leqslant \lambda < 1 \tag{3.3}$$

which assures the existence and uniqueness of the solution of the problem (3.1) [3,4]. Moreover, we obtain $\lim_{\vartheta_u \to 0} \omega(\vartheta_u) = 0$ from (3.2), and by predetermining $\omega(\vartheta_u)$ at zero in continuity, we have

$$\omega(0) = 0 \tag{3.4}$$

If there is a finite section $0 \leqslant \vartheta_u \leqslant \vartheta_s$ of the linearly elastic dependence $\sigma_u = 3G\vartheta_u$ in the material, we have

$$\omega(\vartheta_u) \equiv 0 \quad \text{for} \quad 0 \leqslant \vartheta_u \leqslant \vartheta_s \tag{3.5}$$

We call such materials initially elastic.

In the opposite case, taking account of (3.4) we obtain the asymptotic characteristic

$$\lim_{\vartheta_u \to 0} \vartheta_u \omega'(\vartheta_u) = 0 \tag{3.6}$$

from the requirement of continuity of the first derivative at zero

$$\lim_{\vartheta_u \to 0} \frac{d\sigma_u}{d\vartheta_u} \equiv \lim_{\vartheta_u \to 0} 3G[1 - \omega - \vartheta_u \omega'] = \left.\frac{d\sigma_u}{d\vartheta_u}\right|_{\vartheta_u = 0} \equiv \lim_{\vartheta_u \to 0} \frac{\sigma_u}{\vartheta_u} = 3G$$

Taking account of (3.6), we obtain

$$\lim_{\vartheta_u \to 0} \vartheta_u^2 \omega''(\vartheta_u) = 0 \tag{3.7}$$

from the asymptotic characteristic of the second derivative of the function $\sigma_u(\vartheta_u)$ in the neighborhood of zero,

$$\lim_{\vartheta_u \to 0} \frac{d^2\sigma_u}{d\vartheta_a^2} \vartheta_u \equiv \lim_{\vartheta_u \to 0} 3G[-2\vartheta_u \omega' - \vartheta_u^2 \omega''] = 0$$

etc. As is seen, the asymptotic of the behaviour of the function $\sigma_u(\vartheta_u)$ in the neighborhood of zero defines the asymptotic of the behaviour of the function $\omega(\vartheta_u)$. In the general case, if

$$\lim_{\vartheta_u \to 0} \frac{d^k \sigma_u}{d\vartheta_u^k} \vartheta_u^{k-1} = 0 \quad \text{for} \quad k = 2, 3, \ldots, n \tag{3.8}$$

then

$$\lim_{\partial_u \to 0} \partial_u{}^k \omega^{(k)}(\partial_u) = 0 \quad \text{for} \quad k = 2, 3, \ldots, n \tag{3.9}$$

Materials for which the relationships (3.2), (3.4), (3.6)-(3.9) are satisfied will be called infinitesimally elastic of order n.

It is easy to see that any initially elastic material is infinitesimally elastic of order ∞. In this case, we will assume compliance with conditions (3.2)-(3.4), (3.6), (3.7), and for brevity will call such material infinitesimally elastic.

Let us pose the problem of seeking the process for the change in the form of the external loads for which the process of active homogeneous simple strain will occur in a body.

Let us note that the differentiation operation of any order with respect to time [3,4] can be taken as L in (2.4) for the generating deviator $\partial_{ij}{}^{\circ}(\mathrm{x})$ of the process. Selecting $L = d/dt$, $t_0 = 0$, we obtain from (2.4), (2.6), (2.8)

$$\partial_{ij}{}^{\circ}(\mathrm{x}) \equiv v_{ij}'^{\circ}(\mathrm{x}), \ \varepsilon_{ij}{}^{\circ}(\mathrm{x}), \equiv v_{ij}{}^{\circ}(\mathrm{x}), \ u_i{}^{\circ}(\mathrm{x}) \equiv v_i{}^{\circ}(\mathrm{x}) \tag{3.10}$$

Here $v_i{}^{\circ}(\mathrm{x})$ is the vector of the initial velocities $v_{ij}{}^{\circ} = {}^1\!/_2(v_{i,j}{}^{\circ} + v_{j,i}{}^{\circ})$ and $v_{ij}'^{\circ}$ are the tensor and deviator of the initial strain rates, respectively.

Then the relationships (2.1), (2.2), (2.5), (2.7) describing the kinematics of the homogeneous simple strain process take the respective form

$$\partial_{ij}(\mathrm{x}, t) = A(t) v_{ij}'^{\circ}(\mathrm{x}) \tag{3.11}$$

$$p_{ij} = p_{ij}(\mathrm{x}), \ \partial_u(\mathrm{x}, t) = A(t) v_u{}^{\circ}(\mathrm{x}), \ v_u{}^{\circ} = ({}^2\!/_3 v_{ij}'^{\circ} v_{ij}'^{\circ})^{1/2}$$

$$\varepsilon_{ij}(\mathrm{x}, t) = A(t) v_{ij}{}^{\circ}(\mathrm{x}) + \varepsilon^*(\mathrm{x}, t)\delta_{ij} \tag{3.12}$$

$$u_i(\mathrm{x}, t) = A(t) v_i{}^{\circ}(\mathrm{x}) + u_i{}^*(\mathrm{x}, t)$$

Assuming no initial shear strains ($\partial_{ij}(\mathrm{x}, 0) \equiv 0$), and taking account of the activity of the process, we obtain the following properties of the function $A(t)$ from (3.11)

$$A(0) = 0, \ A^{\cdot}(0) = 1, \ A^{\cdot}(t) \geqslant 0 \quad \text{for} \quad t \in [0, T] \tag{3.13}$$

On the basis of Corollary (2.1), we limit ourselves to the tensor-simple strain process (2.9).

$$\varepsilon_{ij}(\mathrm{x}, t) = A(t) v_{ij}{}^{\circ}(\mathrm{x}), \ u_i(\mathrm{x}, t) = A(t) v_i{}^{\circ}(\mathrm{x}) \tag{3.14}$$

Assuming no initial stresses and strains in the body, and the initial values of the external loads to be zero, we find the necessary and sufficient conditions for the process (3.13), (3.14).

Let the solution of the problem (3.1) be the process (3.13) and (3.14). Then the initial velocities $v_i{}^{\circ}$, the strain rates $v_{ij}{}^{\circ}$, and therefore, the directional tensor p_{ij} also, of the process will be defined uniquely by the initial values of the external load velocities $F_i^{\cdot\circ}(\mathrm{x}) \equiv F_i^{\cdot}(\mathrm{x},\ 0)$, $T_{\nu i}^{\cdot\circ}(\mathrm{x}) \equiv T_{\nu i}^{\cdot}(\mathrm{x},\ 0)$, $\psi_i^{\cdot\circ}(\mathrm{x}) \equiv \psi_i^{\cdot}(\mathrm{x},\ 0)$. Indeed, taking account of (3.4), (3.6), (3.7), (3.11), (3.13) and (3.14), we obtain from (3.1)

$$(3K + G)v_{,i}^{\circ} + G\nabla^2 v_i^{\circ} + F_i^{\cdot\circ} = 0 \text{ in } \quad \Omega \tag{3.15}$$
$$2Gv_{ij}'^{\circ}l_i + 3Kv^{\circ}l_i = T_{\nu i}^{\cdot\circ} \text{ on } S_\sigma, \qquad v_i^{\circ} = \psi_i^{\cdot\circ} \text{ on } S_u$$

The problem (3.15) has the form of a mixed boundary value problem about the static equilibrium of a linearly elastic body with moduli K and G.

The solution $v_i^{\circ}(\mathrm{x})$ of this problem exists and is unique [5] and defines the quantities $v_{ij}^{\circ}(\mathrm{x})$, $p_{ij}(\mathrm{x})$, uniquely, i.e., the "beginning" of the process (3.14). Therefore, the initial velocities of the external loads together with the functions $A(t)$ satisfying conditions (3.13) give the tensor simple strain process (3.14) completely, and the appropriate loading process in addition, i.e., the functions $F_i(\mathrm{x},\ t)$, $T_{\nu i}(\mathrm{x},\ t)$, $\psi_i(\mathrm{x},\ t)$. Using (3.15) we obtain the specific form of the loading process from (3.1),(3.11) and (3.14), namely

$$F_i(\mathrm{x},\ t) = A(t)F_i^{\cdot\circ}(x) - A(t)\left[\omega(\vartheta_u) + \vartheta_u\omega'(\vartheta_u)\right] \times \left[F_i^{\cdot\circ}(\mathrm{x}) + 3Kv_{,i}^{\circ}(\mathrm{x})\right] - 2G\vartheta_u^2\omega'(\vartheta_u)p_{ij,j}(\mathrm{x}) \quad \text{in} \quad Q \tag{3.16}$$

$$T_{\nu i}(\mathrm{x},\ t) = A(t)T_{\nu i}^{\cdot\circ}(\mathrm{x}) - A(t)\omega(\vartheta_u)\left[T_{\nu i}^{\cdot\circ}(\mathrm{x}) - 3Kv^{\circ}(\mathrm{x})l_i\right] \text{ on } \Sigma_\sigma,$$

$$\psi_i(\mathrm{x},\ t) = A(t)\psi_i^{\cdot\circ}(\mathrm{x}) \text{ on } \Sigma_u$$

The right sides of (3.16) are determined uniquely by the function $A(t)$ and the initial external load velocities.

Taking account of (3.15) and (3.11), the relationships (3.16) are necessary, and by virtue of the uniqueness of the solution of problem (3.1), also sufficient conditions of the tensor simple strain process (3.13) and (3.14).

Returning to the problem about the conditions for deviator simple active strain, i.e., turning from (3.14) to (3.12), we superpose an arbitrary quasithermal load field F_i^*, $T_{\nu i}^*$, ψ_i^* on the load field (3.16), according to Corollary 2.1.

T h e o r e m a b o u t h o m o g e n e o u s s i m p l e s t r a i n. In order for the homogeneous simple active strain process (3.11)-(3.13) to occur in a homogeneous, isotropic, infinitesimally (initially) elastic compressible simply-connected body subjected to the external loads F_i, $T_{\nu i}$, ψ_i, it is necessary and sufficient that the following relations be satisfied:

$$F_i(\mathrm{x},\ t) = F_i^t(\mathrm{x},\ t) + F_i^*(t) \quad \text{in } Q \tag{3.17}$$
$$T_{\nu i}(\mathrm{x},\ t) = T^t_{\nu i}(\mathrm{x},\ t) + T_{\nu i}^*(\mathrm{x},\ t) \quad \text{on } \Sigma_\sigma$$
$$\psi_i(\mathrm{x},\ t) = \psi_i^t(\mathrm{x},\ t) + \psi_i^*(\mathrm{x},\ t) \quad \text{on } \Sigma_u$$

Here F_i^t, $T_{\nu i}^t$, ψ_i^t are defined by the relations (3.16) with (3.15) taken into account while F_i^*, $T_{\nu i}^*$, ψ_i^* is an arbitrary quasithermal load field (1.7).

4. Investigation of the homogeneous simple strain

c o n d i t i o n s. A theorem about the homogeneous simple strain for incompressible bodies is formulated analogously with the relation $\sigma \sim \varepsilon$ is replaced by the condition

$$\varepsilon \equiv 0 \quad \text{in} \quad Q' \tag{4.1}$$

The strain tensor is the deviator $\varepsilon_{ij} \equiv \partial_{ij}$ in this case, and the concepts of deviator-simple and tensor-simple strains agree. The quasithermal load field assuring no (shear) strains is determined by the relationships

$$F_i^* = -U_{,i}(\mathbf{x}, t) \quad \text{in} \quad Q' \quad T_{\nu i}^* = [U(\mathbf{x}, t) + D(t)]\, l_i \quad \text{on} \quad \Sigma_\sigma \tag{4.2}$$
$$\psi_i^* = a_i(\mathbf{x}, t) \text{ on } \Sigma_u$$

Here $U(\mathbf{x}, t)$ and $D(t)$ are arbitrary functions, $a_i(\mathbf{x}, t)$ is an arbitrary vector of the displacement of the body as a rigid whole. The appropriate quasithermal solution has the form

$$\sigma_{ij}^* = [U(\mathbf{x}, t) + D(t)]\, \delta_{ij},\ \varepsilon_{ij}^* \equiv 0,\ u_i^* = a_i(\mathbf{x}, t) \tag{4.3}$$

The initial boundary value problem analogous to (3.15) has the form

$$\sigma_{,i}^{\cdot\circ} + G\nabla^2 v_i^\circ + F_i^{\cdot\circ} = 0 \quad \text{in} \quad \Omega, \qquad v_{i,i}^\circ = 0 \quad \text{in} \quad \Omega' \tag{4.4}$$
$$\sigma^{\cdot\circ} l_i + 2G v_{ij}^\circ l_j = T_{\nu i}^{\cdot\circ} \quad \text{on} \quad S_\sigma, \quad v_i^\circ = \psi_i^{\cdot\circ} \text{ on } S_u$$

The loads of the form

$$F_i(\mathbf{x}, t) = A(t) F_i^{\cdot\circ}(\mathbf{x}) - A(t)\,[\omega(\partial_u) + \partial_u \omega'(\partial_u)] \times \tag{4.5}$$
$$[F_i^{\cdot\circ}(\mathbf{x}) + \sigma_{,i}^{\cdot\circ}(\mathbf{x})] - 2G\partial_u^2 \omega'(\partial_u)\, p_{ij,j}(\mathbf{x}) \text{ in } Q$$
$$T_{\nu i}(\mathbf{x}, t) = A(t)\, T_{\nu i}^{\cdot\circ}(\mathbf{x}) - A(t)\,\omega(\partial_u)\,[T_{\nu i}^{\cdot\circ}(\mathbf{x}) - \sigma^{\cdot\circ}(\mathbf{x})\, l_i] \quad \text{on } \Sigma_\sigma$$
$$\psi_i(\mathbf{x}, t) = A(t)\, \psi_i^{\cdot\circ}(\mathbf{x}) \quad \text{on } \Sigma_u$$

are determined uniquely by the functions $A(t)$ satisfying conditions (3.13) and by its initial velocities $F_i^{\cdot\circ}(\mathbf{x})$, $T_{\nu i}^{\cdot\circ}(\mathbf{x})$, $\psi_i^{\cdot\circ}(\mathbf{x})$ with (4.4) taken into account, and assure the homogeneous simple strain process in the body, accompanied by a proportionate growth of the mean stress

$$\varepsilon_{ij}(\mathbf{x}, t) = A(t)\, v_{ij}^\circ(\mathbf{x}), \quad u_i(\mathbf{x}, t) = A(t)\, v_i^\circ(\mathbf{x}), \quad \sigma(\mathbf{x}, t) = A(t)\sigma^{\cdot 0}(\mathbf{x}) \tag{4.6}$$

An arbitrary homogeneous simple strain process is obtained by the linear superposition of the quasithermal process (4.2), (4.3) on the process (4.5), (4.6), from which the necessary and sufficient conditions for homogeneous simple strain of incompressible bodies expressed by (3.17) indeed follow, where the functions (4.5) with (4.4) taken into account should be considered F_i^t, $T_{\nu i}^t$, ψ_i^t while the functions (4.2) should be considered F_i^*, $T_{\nu i}^*$, ψ_i^*

In the case of both compressible and incompressible bodies, the classes of loading processes governed by theorems about homogeneous simple strain are related to the material hardening characteristics, and generally differ substantially from a proportionate change in the external loads. Additional arbitrariness in the change in the external loads is introduced by superposition of quasithermal fields. Proportionate loading is not only a sufficient but also a necessary condition for the homogeneous simple strain of a body to the accuracy of this arbitrariness in particular cases of a linearly elastic material or of a material defined in a simple loading theorem.

The influence of the quasithermal fields on the behaviour of the external loads can turn out to be substantial. For instance, we obtain the continuous, in time, development of the loads F_i^t, T_{vi}^t, ψ_i^t from the zero initial values from the requirement for continuous growth of the shear strains from the zero values. At the same time, the quasithermal field F_i^*, T_{vi}^*, ψ_i^* is certainly not zero at the initial instant and is certainly not generally continuous in time during the process.

Therefore, the results obtained are a foundation for the application of the theory of small elastic-plastic strains for a broader class of elastic-plastic bodies and loads then is provided for in the Il'iushin theorem on simple loading.

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REFERENCES

1. Il'iushin, A. A. Plasticity, Gostekhizdat, Moscow-Leningrad, 1948.

2. Lenskii, V. S., Introduction to Plasticity Theory, Vol. 2, Moscow Univ. Press, 1969.

3. Il'iushin, A. A., Plasticity, USSR Academy of Science Press, Moscow, 1963.

4. Lenskii, V. S. Introduction to Plasticity Theory, Vol. 1, Moscow Univ. Press, 1968.

5. Fichera, G., Existence Theorems in Elasticity Theory "Mir" Moscow, 1974.

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